

Identifying codes of lexicographic product of graphs

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Abstract

Gravier et al. [6] investigated the identifying codes of Cartesian product of two graphs. In this paper we consider the identifying codes of lexicographic product $G[H]$ of a connected graph G and an arbitrary graph H , and obtain the minimum cardinality of identifying codes of $G[H]$ in terms of some parameters of G and H .

Key words: Identifying code; lexicographic product.

1 Introduction

In this paper, we only consider finite undirected simple graphs with at least two vertices. For a given graph G , we often write $V(G)$ for the vertex set of G and $E(G)$ for the edge set of G . For any two vertices u and v of G , $d_G(u, v)$ denotes the distance between u and v in G . Given a vertex $v \in V(G)$, we define $B_G(v) = \{u | u \in V(G), d_G(u, v) \leq 1\}$. A *code* C is a nonempty set of vertices. For a code C , we say that C *covers* v if $B_G(v) \cap C \neq \emptyset$; We say that C *separates* two distinct vertices x and y if $B_G(x) \cap C \neq B_G(y) \cap C$. An *identifying code* of G is a code which covers all the vertices of G and separates any pair of distinct vertices of G . If G admits at least one identifying code, we say G is *identifiable* and denote the minimum cardinality of all identifying codes of G by $I(G)$.

The concept of identifying codes was introduced by Karpovsky et al. [9] to model a fault-detection problem in multiprocessor systems. It was noted in [3, 4] that determining the identifying code with the minimum cardinality in a graph is an NP-complete problem. Many researchers have focused on the study of identifying codes in some restricted classes of graphs, for example, paths [1], cycles [1, 5, 12], and hypercubes [2, 8, 10, 11].

Gravier et al. [6] investigated the identifying codes of Cartesian product of two cliques. In this paper, we consider the identifying codes of lexicographic product $G[H]$ of a connected graph G and an arbitrary graph H . In Section 2, we introduce two new families of codes which are closely related to identifying codes, and compute

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the minimum cardinalities of the two codes for paths and cycles, respectively. In Section 3, we give the sufficient and necessary condition when $G[H]$ is identifiable, and obtain the minimum cardinality of identifying codes of $G[H]$ in terms of some parameters of G and H .

2 Two new families of codes

For a graph H , let $C' \subseteq V(H)$ be a code which separates any pair of distinct vertices of H , we use $I'(H)$ to denote the minimum cardinality of all possible C' ; let $C'' \subseteq V(H)$ be a code which separates any pair of distinct vertices of H and satisfies $C'' \not\subseteq B_H(v)$ for every $v \in V(H)$, we use $I''(H)$ to denote the minimum cardinality of all possible C'' .

The two parameters $I'(H)$ and $I''(H)$ are used to compute the minimum cardinality of identifying codes of $G[H]$ of graphs G and H (see Theorem 3.4). In this section we shall compute the two parameters for paths and cycles, respectively.

Given an integer $n \geq 3$, let P_n be the path of order n and C_n be the cycle of order n . Suppose

$$\begin{aligned} V(P_n) &= \{0, 1, \dots, n-1\}, \quad E(P_n) = \{ij | j = i+1, i = 0, \dots, n-2\}; \\ V(C_n) &= \mathbb{Z}_n = \{0, 1, \dots, n-1\}, \quad E(C_n) = \{ij | j = i+1, i \in \mathbb{Z}_n\}. \end{aligned}$$

Example 1 $I'(P_3) = 2$ and $I''(P_3)$ is not well defined; $I'(P_4) = 3$ and $I''(P_4) = 4$; $I'(P_5) = I''(P_5) = 3$; $I'(P_6) = 3$ and $I''(P_6) = 4$.

For P_4 , $\{0, 1, 2\}$ is an identifying code, but $\{0, 1, 2\} \subseteq B_{P_4}(1)$ and $\{0, 1, 3\}$ can not separate 0 and 1. For P_5 , $\{0, 2, 4\}$ separates any pair of distinct vertices. For P_6 , $\{1, 2, 3\}$ separates any pair of distinct vertices, but $\{1, 2, 3\} \subseteq B_{P_6}(2)$.

Example 2 $I'(C_4) = 3$ and $I''(C_4) = 4$; $I'(C_5) = 3$ and $I''(C_5) = 4$; $I'(C_6) = I''(C_6) = 3$; $I'(C_7) = I''(C_7) = 4$; $I'(C_9) = I''(C_9) = 6$; $I'(C_{11}) = I''(C_{11}) = 6$.

For C_4 , $\{0, 1, 2\}$ is an identifying code, but $\{0, 1, 2\} \subseteq B_{C_4}(1)$. For C_5 , $\{0, 1, 2\}$ is an identifying code, but $\{0, 1, 2\} \subseteq B_{C_5}(1)$ and $\{0, 1, 3\}$ can not separate 0 and 1. For C_6 , both $\{3, 4, 5\}$ and $\{0, 2, 4\}$ separate any pair of distinct vertices. For C_7 , $\{3, 4, 5, 6\}$ separates any pair of distinct vertices. For C_9 , both $\{3, 4, 5, 6, 7, 8\}$ and $\{0, 2, 4, 6, 7, 8\}$ separate any pair of distinct vertices. For C_{11} , $\{3, 4, 5, 8, 9, 10\}$ separates any pair of distinct vertices.

The minimum cardinality of identifying codes of a path or a cycle was computed in [1, 5].

Proposition 2.1 ([1, 5]) (i) For $n \geq 3$, $I(P_n) = \lfloor \frac{n}{2} \rfloor + 1$;

$$(ii) \text{ For } n \geq 6, I(C_n) = \begin{cases} \frac{n}{2}, & n \text{ is even,} \\ \frac{n+3}{2}, & n \text{ is odd.} \end{cases}$$

In order to compute the two parameters for paths and cycle, we need the following useful lemma.

Lemma 2.2 *Let H be an identifiable graph.*

- (i) $I(H) - 1 \leq I'(H) \leq I(H)$;
- (ii) *If $\Delta(H) \leq |V(H)| - 2$, then $I(H) - 1 \leq I'(H) \leq I''(H) \leq I(H) + 1$, where $\Delta(H)$ is the maximum degree of H .*

Proof. Let C' be a code which separates any pair of distinct vertices of H .

(i) Since there exists at most one vertex v not covered by C' , $C' \cup \{v\}$ is an identifying code of H .

(ii) Note that there exists at most one vertex v such that $C' \subseteq B_H(v)$. Since $\Delta(H) \leq |V(H)| - 2$, there exists $v_0 \in V(H) \setminus B_H(v)$ such that $C'' = C' \cup \{v_0\}$ is a code which separates any pair of distinct vertices of H and satisfies $C'' \not\subseteq B_H(w)$ for every $w \in V(H)$. It follows that $I'(H) \leq I''(H) \leq I'(H) + 1$. By (i), (ii) holds. \square

For two integers $i \leq j$, let $[i, j] = \{i, i+1, \dots, j\}$.

Proposition 2.3 *For $n \geq 7$, $I'(P_n) = I''(P_n) = \lfloor \frac{n}{2} \rfloor + 1$.*

Proof. By Lemma 2.2, $I'(P_n) = I(P_n)$ or $I(P_n) - 1$. If $I'(P_n) = I(P_n) - 1$, then there exists a code W' of size $I(P_n) - 1$ such that W' separates any pair of distinct vertices of P_n and $B_{P_n}(i_0) \cap W' = \emptyset$ for a unique $i_0 \in [0, n-1]$.

Case 1. n is odd. Let $W = ([0, i_0] \cap W') \cup \{i-1 \mid i \in [i_0+1, n-1] \cap W'\} \subseteq [0, n-2]$. Since W covers all vertices of P_{n-1} , W is an identifying code of P_{n-1} . By Proposition 2.1,

$$\frac{n+1}{2} = I(P_{n-1}) \leq |W| = |W'| = I(P_n) - 1 = \frac{n-1}{2},$$

a contradiction.

Case 2. n is even. By Proposition 2.1, $|W'| = I(P_n) - 1 = \frac{n}{2}$.

Case 2.1. $i_0 \neq 0$ and $i_0 \neq n-1$. Then $i_0 - 1, i_0, i_0 + 1 \notin W'$, and $i_0 - 2, i_0 - 3, i_0 - 4, i_0 + 2, i_0 + 3, i_0 + 4 \in W'$, so $4 \leq i_0 \leq n-5$. Let $W = W' \cap [0, i_0 - 1]$ and $\overline{W} = \{i - i_0 - 1 \mid i \in W' \cap [i_0 + 1, n-1]\}$. Then W is an identifying code of P_{i_0} and \overline{W} is an identifying code of P_{n-i_0-1} . By Proposition 2.1, we have

$$\frac{n}{2} = |W'| = |W| + |\overline{W}| \geq I(P_{i_0}) + I(P_{n-i_0-1}) = \lfloor \frac{i_0}{2} \rfloor + 1 + \lfloor \frac{n-i_0-1}{2} \rfloor + 1 = \frac{n+2}{2},$$

a contradiction.

Case 2.2. $i_0 = 0$ or $n-1$. Without loss of generality, assume $i_0 = n-1$. Then $n-1, n-2 \notin W'$, and $n-3, n-4, n-5 \in W'$. We can observe the following results:

$$|W' \cap [i, i+3]| \geq 2, i \in [0, n-4], \quad (1)$$

$$|W' \cap [0, 2]| \geq 2, \quad (2)$$

$$|W' \cap [0, 4]| \geq 3. \quad (3)$$

Case 2.2.1. $n = 4k$. By (1) and (2), $2k = |W'| \geq 2\lfloor \frac{n-5-3}{4} \rfloor + 3 + 2 = 2k + 1$, a contradiction.

Case 2.2.2. $n = 4k+2$. By (1) and (3), $2k+1 = |W'| \geq 2\lfloor \frac{n-5-5}{4} \rfloor + 3 + 3 = 2k+2$, a contradiction.

Therefore, $I'(P_n) = I(P_n)$. Note that $I''(P_n) = I'(P_n)$ when $I'(P_n) \geq 4$. By Proposition 2.1, the desired result follows. \square

Proposition 2.4 $I'(C_n) = I''(C_n) = \begin{cases} \frac{n}{2}, & n \text{ is even and } n \geq 8, \\ \frac{n+3}{2}, & n \text{ is odd and } n \geq 13. \end{cases}$

Proof. If $I'(C_n) < \lceil \frac{n}{2} \rceil$, then there exists a code W' such that $|W'| = I'(C_n) < \lceil \frac{n}{2} \rceil$ and W' separates any pair of distinct vertices of C_n . It follows that there exists i_0 such that $i_0, i_0 + 1 \notin W'$. Without loss of generality, assume $n - 1, 0 \notin W'$. Since W' is also a subset of $V(P_n)$ and $B_{C_n}(j) \cap W' = B_{P_n}(j) \cap W'$ for any $j \in [0, n - 1]$, W' separates any pair of distinct vertices of P_n . By Proposition 2.3, $\lceil \frac{n}{2} \rceil \leq I'(P_n) \leq |W'| < \lceil \frac{n}{2} \rceil$, a contradiction. Hence $I'(C_n) \geq \lceil \frac{n}{2} \rceil$.

Case 1. n is even and $n \geq 8$. By Proposition 2.1 and Lemma 2.2, $\frac{n}{2} \leq I'(C_n) \leq I(C_n) = \frac{n}{2}$. Hence $I'(C_n) = \frac{n}{2}$.

Case 2. n is odd and $n \geq 13$. By Proposition 2.1 and Lemma 2.2, $I'(C_n) = \frac{n+3}{2}$ or $\frac{n+1}{2}$. If $I'(C_n) = \frac{n+1}{2}$, then there exists a code W' of size $\frac{n+1}{2}$ such that W' separates any pair of distinct vertices of C_n and $B_{C_n}(i_0) \cap W' = \emptyset$ for a unique $i_0 \in [0, n - 1]$. Without loss of generality, assume $i_0 = 1$. Then $0, 1, 2 \notin W'$ and $3, 4, 5, n - 3, n - 2, n - 1 \in W'$. We can observe the following results:

$$|W' \cap [i, i + 3]| \geq 2, i \in [6, n - 7], \quad (4)$$

$$|W' \cap [6, 11]| \geq 3. \quad (5)$$

Case 2.2.1. $n = 4k + 1$. By (4), $2k + 1 = |W'| \geq 2\lfloor \frac{n-9}{4} \rfloor + 6 = 2k + 2$, a contradiction.

Case 2.2.2. $n = 4k + 3$. By (4) and (5), $2k + 2 = |W'| \geq 2\lfloor \frac{n-9-6}{4} \rfloor + 6 + 3 = 2k + 3$, a contradiction.

Therefore, $I'(C_n) = \frac{n+3}{2}$.

Since $I''(C_n) = I'(C_n)$ when $I'(C_n) \geq 4$, the desired result follows. \square

3 Main results

The *lexicographic product* $G[H]$ of graphs G and H is the graph with the vertex set $V(G) \times V(H) = \{(u, v) | u \in V(G), v \in V(H)\}$, and the edge set $\{((u_1, v_1), (u_2, v_2)) | d_G(u_1, u_2) = 1, \text{ or } u_1 = u_2 \text{ and } d_H(v_1, v_2) = 1\}$. For any two distinct vertices $(u_1, v_1), (u_2, v_2)$ of $G[H]$, we observe that

$$d_{G[H]}((u_1, v_1), (u_2, v_2)) = \begin{cases} 1, & \text{if } u_1 = u_2, d_H(v_1, v_2) = 1, \\ 2, & \text{if } u_1 = u_2, d_H(v_1, v_2) \geq 2, \\ d_G(u_1, u_2), & \text{if } u_1 \neq u_2. \end{cases} \quad (6)$$

For $u \in V(G)$, let $N_G(u) = B_G(u) \setminus \{u\}$. For any $u_1, u_2 \in V(G)$, define $u_1 \equiv u_2$ if and only if $B_G(u_1) = B_G(u_2)$ or $N_G(u_1) = N_G(u_2)$. Hernando et al. [7] proved that “ \equiv ” is an equivalent relation and the equivalence class of a vertex is of three types: a class of size 1, a clique of size at least 2, an independent set of size at least 2. Denote all equivalence classes by

$$W_1, \dots, W_p, U_1, \dots, U_k, V_1, \dots, V_l, \quad (7)$$

where

- (i) $|W_q| = 1, q = 1, \dots, p$;
- (ii) for any $u_1, u_2 \in U_i, i = 1, \dots, k, B_G(u_1) = B_G(u_2)$;
- (iii) for any $u_1, u_2 \in V_j, j = 1, \dots, l, N_G(u_1) = N_G(u_2)$.

Denote $s(G) = |U_1| + \dots + |U_k| - k, t(G) = |V_1| + \dots + |V_l| - l$. We give an algorithm of computing $s(G)$ and $t(G)$ in Appendix.

For $u \in V(G)$ and $C \subseteq V(H)$, let $C^u = \{(u, v) | (u, v) \in V(G[H]), v \in C\}$. For $S \subseteq V(G[H])$, let $S_u = \{v | v \in V(H), (u, v) \in S\}$. Note that $(S_u)^u = H^u \cap S$, where $H^u = (V(H))^u$. By (6), we have

$$B_{G[H]}((u, v)) = (B_H(v))^u \cup \bigcup_{w \in N_G(u)} H^w, \quad (8)$$

$$B_{G[H]}((u, v)) \cap S = ((B_H(v)) \cap S_u)^u \cup \bigcup_{w \in N_G(u)} (S_w)^w. \quad (9)$$

In the rest of this section we always assume that G is a connected graph and H is an arbitrary graph.

Theorem 3.1 *The lexicographic product $G[H]$ of graphs G and H is identifiable if and only if*

- (i) H is identifiable and $\Delta(H) \leq |V(H)| - 2$, or
- (ii) both G and H are identifiable.

Proof. Suppose $G[H]$ is identifiable. If H is not identifiable, then there exist two distinct vertices v_1, v_2 of H with $B_H(v_1) = B_H(v_2)$. By (8), $B_{G[H]}((u, v_1)) = B_{G[H]}((u, v_2))$ for $u \in V(G)$. This contradicts the condition that $G[H]$ is identifiable.

If $\Delta(H) = |V(H)| - 1$ and G is not identifiable, then there exist $v \in V(H)$ and two distinct vertices u_1, u_2 of G such that

$$B_H(v) = V(H) \text{ and } B_G(u_1) = B_G(u_2).$$

By (8), we have

$$B_{G[H]}((u_1, v)) = H^{u_1} \cup \bigcup_{u \in N_G(u_1)} H^u = \bigcup_{u \in B_G(u_1)} H^u = \bigcup_{u \in B_G(u_2)} H^u = B_{G[H]}((u_2, v)).$$

This contradicts the condition that $G[H]$ is identifiable.

Therefore, (i) or (ii) holds.

Conversely, suppose (i) or (ii) holds. Assume that $G[H]$ is not identifiable. Therefore, there exist two distinct vertices $(u_1, v_1), (u_2, v_2)$ such that $B_{G[H]}((u_1, v_1)) = B_{G[H]}((u_2, v_2))$. If $u_1 \neq u_2$, then $d_G(u_1, u_2) = 1$. It follows that $B_G(u_1) = B_G(u_2)$ and $B_H(v_1) = B_H(v_2) = V(H)$, contrary to (i) and (ii). If $u_1 = u_2$, then $v_1 \neq v_2$. By (8), $B_H(v_1) = B_H(v_2)$, contrary to the condition that H is identifiable. \square

Remark. Let r be a positive integer and Γ be a graph. Given a vertex $v \in V(\Gamma)$, define $B_\Gamma^{(r)}(v) = \{u | u \in V(\Gamma), d_\Gamma(u, v) \leq r\}$. An r -identifying code of Γ is a code which r -covers all the vertices of Γ and r -separates any pair of distinct vertices of Γ (see [9] for details). Identifying codes in this paper are 1-identifying codes. If $r \geq 2$, then $G[H]$ does not admit any r -identifying code. Indeed, by (6), $B_{G[H]}^{(r)}((u, v_1)) = B_{G[H]}^{(r)}((u, v_2))$ for $r \geq 2$.

Lemma 3.2 *If S is an identifying code of $G[H]$, then for any vertex u of G , S_u separates any pair of distinct vertices of H . Moreover, with reference to (7),*

- (i) *if $k \neq 0$, then there exists at most one vertex $u \in U_i$ satisfying $S_u \subseteq B_H(v)$ for a vertex v of H , where $i = 1, \dots, k$;*
- (ii) *if $l \neq 0$, then there exists at most one vertex $u \in V_j$ satisfying $S_u \cap B_H(v) = \emptyset$ for a vertex v of H , where $j = 1, \dots, l$.*

Proof. Assume that there exist $u_0 \in V(G)$ and two distinct vertices v_1, v_2 of H such that $S_{u_0} \cap B_H(v_1) = S_{u_0} \cap B_H(v_2)$. By (9), $B_{G[H]}((u_0, v_1)) \cap S = B_{G[H]}((u_0, v_2)) \cap S$, contrary to the condition that S is an identifying code of $G[H]$.

(i) Assume that there exist two distinct vertices $u_1, u_2 \in U_i$ such that $S_{u_1} \subseteq B_H(v_1)$ and $S_{u_2} \subseteq B_H(v_2)$. Since $B_G(u_1) = B_G(u_2)$, by (9) we have

$$B_{G[H]}((u_1, v_1)) \cap S = (S_{u_1})^{u_1} \cup \bigcup_{u \in N_G(u_1)} (S_u)^u = \bigcup_{u \in B_G(u_2)} (S_u)^u = B_{G[H]}((u_2, v_2)) \cap S.$$

Since S is an identifying code of $G[H]$, $(u_1, v_1) = (u_2, v_2)$, a contradiction.

(ii) Assume that there exist two different vertices $u_1, u_2 \in V_j$ such that $S_{u_1} \cap B_H(v_1) = S_{u_2} \cap B_H(v_2) = \emptyset$. Since $N_G(u_1) = N_G(u_2)$, by (9) we have

$$B_{G[H]}((u_1, v_1)) \cap S = \bigcup_{u \in N_G(u_1)} (S_u)^u = \bigcup_{u \in N_G(u_2)} (S_u)^u = B_{G[H]}((u_2, v_2)) \cap S.$$

Since S is an identifying code of $G[H]$, $(u_1, v_1) = (u_2, v_2)$, a contradiction. \square

In equivalence classes (7) of $V(G)$, choose $\bar{u}_i \in U_i, i = 1, \dots, k$, and $\bar{v}_j \in V_j, j = 1, \dots, l$. Let $\bar{W}_0 = \cup_{q=1}^p W_q \cup \{\bar{u}_1, \dots, \bar{u}_k, \bar{v}_1, \dots, \bar{v}_l\}$ and $\bar{U}_i = U_i \setminus \{\bar{u}_i\}, i = 1, \dots, k$, $\bar{V}_j = V_j \setminus \{\bar{v}_j\}, j = 1, \dots, l$. Therefore, we have a partition of $V(G)$:

$$\bar{W}_0, \bar{U}_1, \dots, \bar{U}_k, \bar{V}_1, \dots, \bar{V}_l. \quad (10)$$

Lemma 3.3 *Let C be an identifying code of graph H , and let C', C'' be two codes which separate any pair of distinct vertices of H and $C'' \not\subseteq B_H(v)$ for every vertex v of H . With reference to (10),*

$$S = \bigcup_{u \in \bar{W}_0} (C')^u \cup \bigcup_{i=1}^k \bigcup_{u \in \bar{U}_i} (C'')^u \cup \bigcup_{i=1}^l \bigcup_{u \in \bar{V}_i} C^u$$

is an identifying code of $G[H]$.

Proof. For any $u \in V(G)$, we have

$$S_u = \begin{cases} C', & \text{if } u \in \bar{W}_0, \\ C'', & \text{if } u \in \cup_{i=1}^k \bar{U}_i, \\ C, & \text{if } u \in \cup_{j=1}^l \bar{V}_j. \end{cases}$$

Since G is connected, there exists a vertex w adjacent to u . By (6), S covers all vertices of $G[H]$. For any two distinct vertices $(u_1, v_1), (u_2, v_2) \in V(G[H])$, we only need to show that

$$B_{G[H]}((u_1, v_1)) \cap S \neq B_{G[H]}((u_2, v_2)) \cap S. \quad (11)$$

To prove (11), it is sufficient to show that there exists $(u_0, v_0) \in S$ such that

$$d_{G[H]}((u_0, v_0), (u_1, v_1)) \leq 1, \quad d_{G[H]}((u_0, v_0), (u_2, v_2)) \geq 2 \quad (12)$$

or

$$d_{G[H]}((u_0, v_0), (u_2, v_2)) \leq 1, \quad d_{G[H]}((u_0, v_0), (u_1, v_1)) \geq 2. \quad (13)$$

Case 1. $u_1 \neq u_2$. Then there exists $u_0 \in V(G) \setminus \{u_1, u_2\}$ such that $d_G(u_1, u_0) = 1$ and $d_G(u_2, u_0) \geq 2$, or $d_G(u_1, u_0) \geq 2$ and $d_G(u_2, u_0) = 1$. Take $v_0 \in S_{u_0}$. Then $(u_0, v_0) \in S$. By (6), (12) or (13) holds.

Case 2. $u_1 \equiv u_2$.

Case 2.1. $u_1 = u_2$. Since S_{u_1} separates v_1 and v_2 , $B_H(v_1) \cap S_{u_1} \neq B_H(v_2) \cap S_{u_1} = B_H(v_2) \cap S_{u_2}$. By (9), (11) holds.

Case 2.2. $u_1 \neq u_2$ and $B_G(u_1) = B_G(u_2)$. Then u_1 and u_2 are adjacent and fall into some U_i . It follows that $u_1 \in \overline{U}_i$ or $u_2 \in \overline{U}_i$. Without loss of generality, suppose $u_1 \in \overline{U}_i$. Pick $u_0 = u_1$. Since $C'' \not\subseteq B_H(v_1)$, there exists $v_0 \in C''$ such that $(u_0, v_0) \in S$ and $d_H(v_0, v_1) \geq 2$. By (6), (13) holds.

Case 2.3. $u_1 \neq u_2$ and $N_G(u_1) = N_G(u_2)$. Then u_1 and u_2 are at distance 2 and fall into some V_j . It follows that $u_1 \in \overline{V}_j$ or $u_2 \in \overline{V}_j$. Without loss of generality, suppose $u_1 \in \overline{V}_j$. Pick $u_0 = u_1$. Since C covers v_1 , there exists $v_0 \in C$ such that $(u_0, v_0) \in S$ and $d_H(v_0, v_1) \leq 1$. By (6), (12) holds. \square

Theorem 3.4 Suppose (i) or (ii) holds in Theorem 3.1.

(i) If $\Delta(H) \leq |V(H)| - 2$, then

$$I(G[H]) = (|V(G)| - s(G) - t(G))I'(H) + s(G)I''(H) + t(G)I(H); \quad (14)$$

(ii) If $\Delta(H) = |V(H)| - 1$, then

$$I(G[H]) = (|V(G)| - t(G))I'(H) + t(G)I(H). \quad (15)$$

Proof. (i) By Theorem 3.1, $I(H)$ and $I'(H)$ are well defined. Since $V(H)$ separates any pair of distinct vertices of H and $V(H) \not\subseteq B_H(v)$ for every $v \in V(H)$, $I''(H)$ is well defined.

Let S be an identifying code of $G[H]$ with the minimum cardinality, by Lemma 3.2,

$$\begin{aligned} I(G[H]) &= |S| = \sum_{i=1}^p \sum_{u \in W_i} |S_u| + \sum_{i=1}^k \sum_{u \in U_i} |S_u| + \sum_{i=1}^l \sum_{u \in V_i} |S_u| \\ &\geq (p + k + l)I'(H) + (\sum_{i=1}^k |U_i| - l)I''(H) + (\sum_{i=1}^l |V_i| - l)I(H) \\ &= (|V(G)| - s(G) - t(G))I'(H) + s(G)I''(H) + t(G)I(H). \end{aligned}$$

Let C be an identifying code of H with the minimum cardinality. Let C' and C'' be two codes with the minimum cardinality such that they separate any pair of distinct vertices of H and $C'' \not\subseteq B_H(v)$ for every vertex v of H . By Lemma 3.3,

$$I(G[H]) \leq |S| = (|V(G)| - s(G) - t(G))I'(H) + s(G)I''(H) + t(G)I(H).$$

Therefore, (14) holds.

(ii) By Theorem 3.1, both G and H are identifiable. So $I(H)$ and $I'(H)$ are well defined. Owing to $B_G(u_1) \neq B_G(u_2)$ for any two distinct vertices u_1, u_2 of G , we get $k = 0$ in (7) and (10). Similar to the proof of (i), (15) holds. \square

Combining Propositions 2.1, 2.3, 2.4 and Theorem 3.4, we have

Corollary 3.5 *Let G be a connected graph of order m ($m \geq 2$).*

- (i) *For $n \geq 7$, $I(G[P_n]) = m(\lfloor \frac{n}{2} \rfloor + 1)$;*
- (ii) *For $n \geq 12$, $I(G[C_n]) = \begin{cases} \frac{mn}{2}, & n \text{ is even,} \\ \frac{m(n+3)}{2}, & n \text{ is odd.} \end{cases}$*

Appendix

Algorithm	
Input	Graph G
Output	$W_1, \dots, W_p, U_1, \dots, U_k, V_1, \dots, V_l$ //the equivalent classes of $V(G)$ $s(G), t(G)$
Step 1.	Preparation//Input the adjacent matrix A of G and $A + E$ (E is an identity matrix).
1.	$V(G) = \{1, \dots, m\}; E(G) = \{ij ij \text{ are adjacent in } G\}$
2.	for $i = 1, \dots, m$ do
3.	for $j = 1, \dots, m$ do
4.	if $j = i$ then $a_{ij} := 0$ and $\bar{a}_{ij} := 1$
5.	else if $ij \in E$ then $a_{ij} := 1$ and $\bar{a}_{ij} := 1$
6.	else $a_{ij} := 0$ and $\bar{a}_{ij} := 0$
7.	end-if
8.	end-if
9.	end-for
10.	end-for
11.	for $i = 1, \dots, m$ do
12.	$A_i := (a_{i1}, \dots, a_{im}); \bar{A}_i := (\bar{a}_{i1}, \dots, \bar{a}_{im})$
13.	end-for
Step 2.	Output the equivalent classes of $V(G)$
14.	$i := 1; p := 1; k := 1; l := 1; I := \emptyset$
15.	while $i \leq m$ do
16.	if $i \in I$ then $i := i + 1$ // $i \equiv i_0$ for some $i_0 < i$
17.	else if $i \leq m - 1$ then $W_p = \{i\}; U_k = \{i\}; V_l = \{i\}$ and do
18.	for $j = i + 1, \dots, m$ do
19.	if $\bar{A}_j = \bar{A}_i$ then $I := I \cup \{j\}$ and $U_k := U_k \cup \{j\}$ // $B_G(j) = B_G(i)$
20.	else if $A_j = A_i$ then $I := I \cup \{j\}$ and $V_l := V_l \cup \{j\}$ // $N_G(j) = N_G(i)$
21.	end-if
22.	end-for
23.	if $ U_k > 1$ then output U_k and $k := k + 1$
24.	else if $ V_l > 1$ then output V_l and $l := l + 1$
25.	else output W_p and $p := p + 1$ // $i \not\equiv j$ for any $j \in V(G)$
26.	end-if
27.	end-if
28.	$i := i + 1$
29.	else $W_p := \{i\}$, $i := i + 1$ and output W_p
30.	end-if
31.	end-if
32.	end-while
Step 3.	Compute $s(G)$ and $t(G)$
33.	$s = 0; t = 0$
34.	If $k > 1$ then
35.	for $i = 1, \dots, k - 1$ do
36.	$s := s + U_i $
37.	end-for
38.	If $k > 1$ then
39.	for $i = 1, \dots, k - 1$ do
40.	$t := t + V_i $
41.	end-for
42.	output $s(G) = s$ and $t(G) = t$

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